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Irreducible groups with submultiplicative spectrum[☆]

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Abstract

Let $\mathscr{G}$ be a finite group of matrices. The spectrum σ is *submultiplicative* on $\mathscr{G}$ if

$$\sigma(ST) \subseteq \sigma(S)\sigma(T) = \{\lambda\mu : \lambda \in \sigma(S), \mu \in \sigma(T)\}$$

for every $S, T \in \mathscr{G}$. We construct examples of irreducible groups of matrices with submultiplicative spectrum over vector spaces of all possible finite dimensions. We show that in the case of even dimensions the divisibility by 8 is required for irreducibility of the group.

Keywords: Matrix groups; Irreducible groups; Submultiplicative spectrum

1. Introduction

Let $\mathscr{S}$ be a multiplicative semigroup of operators on a complex Hilbert space. We call $\mathscr{S}$ *reducible*, if it has a nontrivial invariant subspace, and *irreducible* otherwise. The spectrum σ is *submultiplicative* on $\mathscr{S}$ if

$$\sigma(ST) \subseteq \sigma(S)\sigma(T) = \{\lambda\mu : \lambda \in \sigma(S), \mu \in \sigma(T)\}$$

for every $S, T \in \mathscr{S}$. Our aim is to study irreducible semigroups of operators with submultiplicative spectrum.

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The connection between reducibility of a semigroup and submultiplicativity of its spectrum has already been studied. An example of irreducible group with submultiplicative spectrum was first presented by Lambrou–Longstaff–Radjavi [2]; it was shown that for every odd integer n there exists a semigroup of the kind consisting of operators on an n-dimensional space; it was also mentioned that in the case $n = 2$ such a semigroup does not exist. The results of this paper were included and strengthened in [4], but the question of existence of semigroups of the kind consisting of operators on n-dimensional space for even n, $n \neq 2$, was left unsolved. Omladič [3] studied 2-groups with submultiplicative spectrum and gave an example of such a group of operators on vector space of dimension 8, which is also irreducible. However, if $\mathscr{S}$ is a semigroup of compact operators on an infinite dimensional space and spectrum is submultiplicative on $\mathscr{S}$, then $\mathscr{S}$ is always reducible (cf. [4, Theorem 8.3.5]).

We are interested in finding irreducible semigroups with submultiplicative spectrum of operators on vector spaces of all possible finite dimensions. We shall explicitly construct two examples and prove the existence of such a group on n-dimensional space for every even n which is divisible by 8. We shall also see that the assumption on divisibility by 8 is necessary. Finally we will show that the constructed examples are minimal possible.

Let us first summarize some results on irreducible semigroups with submultiplicative spectrum. If $\mathscr{S}$ is a semigroup of the kind, then $\mathscr{S}\backslash\{0\}$ is a *group* which is *essentially finite*, i.e. $\mathscr{S} \subseteq \mathbb{C}\mathscr{G}_0$ for some finite group $\mathscr{G}_0$ (cf. [4, Theorem 3.3.4]). This allows us to restrict our study from semigroups to finite groups. Also, our groups are nilpotent:

Proposition 1. *An irreducible group with submultiplicative spectrum is direct product of its Sylow p-subgroups* (*cf.* [4, *Theorem* 3.3.5]).

Furthermore, these Sylow p-subgroups are irreducible themselves:

Proposition 2. *The direct product of two groups is irreducible if and only if it is isomorphic to direct product of irreducible groups* (*cf.* [6, § 3.2, *Theorem* 10]).

We shall identify operators with their matrices relative to a fixed basis. If A and B are square matrices of size m and n, respectively, then the tensor product $A \otimes B$ is a square matrix of size mn, defined as

$$A \otimes B := \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mm}B \end{pmatrix}, \quad \text{where} \quad A = (a_{ij})_{i,j=1}^{m}.$$

For semigroups of matrices $\mathscr{S}$ and $\mathscr{T}$ we will denote

$$\mathscr{S} \otimes \mathscr{T} := \{S \otimes T : S \in \mathscr{S}, T \in \mathscr{T}\}.$$

Proposition 3. *If $\mathscr{S}$ and $\mathscr{T}$ are irreducible semigroups with submultiplicative spectrum, then $\mathscr{S} \otimes \mathscr{T}$ is also an irreducible semigroup with submultiplicative spectrum* (*cf*. [4, *p*. 60]).

2. Construction

We start by construction of two irreducible groups of matrices with submultiplicative spectrum, one acting on the vector space of dimension 16 and the other on the vector space of dimension 32. We shall denote these two groups by ${}^{16}\mathscr{G}$ and ${}^{32}\mathscr{G}$, respectively. The construction of groups ${}^{16}\mathscr{G}$ and ${}^{32}\mathscr{G}$ presented here is based on the example of 8×8 matrices given in [3]. We denote the group constructed in that example by ${}^{8}\mathscr{G}$. The groups ${}^{8}\mathscr{G}$, ${}^{16}\mathscr{G}$, and ${}^{32}\mathscr{G}$ help us to obtain all possible even-dimensional examples of irreducible groups with submultiplicative spectrum.

For a group $\mathscr{G}$ with exponent 2^{s+1} we denote by $\mathscr{G}_k$ the set of all elements of $\mathscr{G}$ with order no more than 2^k. If $\mathscr{G}$ has submultiplicative spectrum we get a normal series:

$$\begin{gathered}\{I\} = \mathscr{G}_0 \lhd \mathscr{G}_1 \lhd \cdots \lhd \mathscr{G}_s \lhd \mathscr{G}_{s+1} = \mathscr{G}, \\ \text{where } \mathscr{G}_k \text{ contains the square of every element of } \mathscr{G}_{k+1},\ 0 \leqslant k \leqslant s\end{gathered} \tag{1}$$

(cf. [3, Lemma 4.3]). Note that this property is inherited by restrictions to invariant subspaces while the submultiplicativity of spectrum is not.

Since our groups are finite and nilpotent, we can assume that they are monomial (cf. [6, § 8.5, Theorem 16]). In the case of irreducible group $\mathscr{G}$ the following lemma gives us also block-monomiality and block-diagonality of groups in (1):

Lemma 4 [3, Lemma 4.1]. *Let $\mathscr{G}$ be an irreducible group with a subgroup $\mathscr{H}$ that contains the square of every element of $\mathscr{G}$. Then, $\mathscr{G}$ is block-monomial with respect to the block-partition in which $\mathscr{H}$ is block-diagonal with irreducible diagonal blocks.*

These observations provide the main idea of our constriction of matrices. We shall construct them blockwise, level by level, to obtain a fractal-like structure.

In construction we shall use special matrix operation which we define for 2-*matrices* (i.e. square matrices of size 2^s for some $s \in \mathbb{N}$). This operation is *amplification*, defined on different block-levels for a given 2-matrix. If A is a square matrix of size 2^s, then on kth level, $k = 0, \ldots, s,$ A can be viewed as a $2^k \times 2^k$ block matrix with blocks of size 2^{s-k}. For such a matrix $A = (A_{ij})_{i,j=1}^{2^k}$ the matrix $A^{(2,k)}$ is a square matrix of size 2^{s+1} and is defined as

$$A^{(2,0)} = A^{(2)} := \begin{pmatrix} A & \\ & A \end{pmatrix} \quad \text{and}$$

$$A^{(2,k)} := \begin{pmatrix} A_{11}^{(2)} & \cdots & A_{12^k}^{(2)} \\ \vdots & \ddots & \vdots \\ A_{2^k1}^{(2)} & \cdots & A_{2^k2^k}^{(2)} \end{pmatrix}, \quad 1 \leqslant k \leqslant s.$$

We follow the notation of [3, § 5]:

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2}$$

Note that the squares of all these matrices are diagonal: $d^2 = f^2 = e$, $g^2 = -e$ and that the set $\{\pm e, \pm d, \pm f, \pm g\}$ is an irreducible multiplicative group of exponent 4. From these matrices Omladič in [3, § 5] constructed an irreducible group ${}^8\mathscr{G}$ of exponent 4 with submultiplicative spectrum. The generators of the group ${}^8\mathscr{G}$ are:

$$X = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 0 & g \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & f & 0 & 0 \\ f & 0 & 0 & 0 \\ 0 & 0 & 0 & f \\ 0 & 0 & -f & 0 \end{pmatrix},$$

$$Z = \begin{pmatrix} 0 & 0 & f & 0 \\ 0 & 0 & 0 & g \\ -g & 0 & 0 & 0 \\ 0 & -f & 0 & 0 \end{pmatrix}.$$

Their squares

$$U = X^2 = (d \otimes e)^{(2)}, \quad V = Y^2 = d \otimes e^{(2)}, \quad W = Z^2 = d \otimes d \otimes d.$$

generate a commutative subgroup of diagonal matrices while X, Y and Z are block-diagonal on different levels. We shall construct our examples by 'blowing up' these matrices with the operation defined above.

Let us first consider the case of the 16×16 matrices. Introduce:

$$\dot{X} := X^{(2)}, \quad \dot{Y} := Y^{(2)}, \quad \dot{Q} := Y^{(2,2)}, \quad \dot{Z} := Z^{(2,1)}.$$

Then

$$\dot{U} := \dot{X}^2 = U^{(2)}, \quad \dot{V} := \dot{Y}^2 = V^{(2)}, \quad \dot{T} := \dot{Q}^2 = V^{(2,1)}, \quad \dot{W} := W^{(2,1)}.$$

Matrices $\dot{U}$, $\dot{V}$, $\dot{T}$, and $\dot{W}$ are diagonal and therefore commute with each other. We check at once that their 'square roots' $\dot{X}$, $\dot{Y}$, $\dot{Q}$, and $\dot{Z}$ either commute or anti-commute with them:

$$\begin{array}{llll} \dot{X}\dot{U} = \dot{U}\dot{X}, & \dot{X}\dot{V} = \dot{V}\dot{X}, & \dot{X}\dot{T} = \dot{T}\dot{X}, & \dot{X}\dot{W} = -\dot{W}\dot{X}, \\ \dot{Y}\dot{U} = -\dot{U}\dot{Y}, & \dot{Y}\dot{V} = \dot{V}\dot{Y}, & \dot{Y}\dot{T} = \dot{T}\dot{Y}, & \dot{Y}\dot{W} = \dot{W}\dot{Y}, \\ \dot{Q}\dot{U} = \dot{U}\dot{Q}, & \dot{Q}\dot{V} = -\dot{V}\dot{Q}, & \dot{Q}\dot{T} = \dot{T}\dot{Q}, & \dot{Q}\dot{W} = -\dot{W}\dot{Q}, \\ \dot{Z}\dot{U} = \dot{U}\dot{Z}, & \dot{Z}\dot{V} = \dot{V}\dot{Z}, & \dot{Z}\dot{T} = -\dot{T}\dot{Z}, & \dot{Z}\dot{W} = \dot{W}\dot{Z}. \end{array} \tag{3}$$

On the other hand, an easy computation shows that the following commuting relations hold between $\dot{X}$, $\dot{Y}$, $\dot{Q}$, and $\dot{Z}$:

$$\begin{array}{lll} \dot{X}\dot{Y} = -\dot{T}\dot{W}\dot{Y}\dot{X}, & \dot{X}\dot{Q} = \dot{U}\dot{Q}\dot{X}, & \dot{X}\dot{Z} = \dot{T}\dot{Z}\dot{X}, \\ \dot{Y}\dot{Q} = \dot{U}\dot{Q}\dot{Y}, & \dot{Y}\dot{Z} = -\dot{W}\dot{Z}\dot{Y}, & \dot{Q}\dot{Z} = \dot{T}\dot{U}\dot{V}\dot{Z}\dot{Q}. \end{array} \tag{4}$$

Let ${}^{16}\mathscr{G}$ be the group generated by matrices $\dot{X}$, $\dot{Y}$, $\dot{Q}$, and $\dot{Z}$. The relations (3) and (4) show us that every element of ${}^{16}\mathscr{G}$ could be uniquely expressed as a word in letters $\dot{U}$, $\dot{V}$, $\dot{T}$, $\dot{W}$, $\dot{X}$, $\dot{Y}$, $\dot{Q}$, and $\dot{Z}$ (in this order), with possible negative sign. In particular, ${}^{16}\mathscr{G}$ is a finite group with $2^9 = 512$ elements.

Denote by ${}^{16}\mathscr{D}$ the set of all diagonal matrices from ${}^{16}\mathscr{G}$ and by ${}^{16}\mathscr{D}^0$ the subset of all elements of ${}^{16}\mathscr{D}$ of trace zero. Clearly, ${}^{16}\mathscr{D}$ is a commutative group generated by matrices $\dot{U}$, $\dot{V}$, $\dot{T}$, $\dot{W}$, and $-I$. Observe that ${}^{16}\mathscr{D}^0$ consists of diagonal matrices with 8 entries equal to 1 and 8 entries equal to -1. Group ${}^{16}\mathscr{D}$ contains ${}^{16}\mathscr{D}^0$ together with the matrices I and $-I$. We begin with some properties of the nondiagonal matrices of the group ${}^{16}\mathscr{G}$.

Lemma 5. *The square of every nondiagonal element of* ${}^{16}\mathscr{G}$ *belongs to* ${}^{16}\mathscr{D}^0$.

Proof. We show this by expressing each element of ${}^{16}\mathscr{G}\backslash{}^{16}\mathscr{D}$ as a word in letters $\dot{U}$, $\dot{V}$, $\dot{T}$, $\dot{W}$, $\dot{X}$, $\dot{Y}$, $\dot{Q}$, and $\dot{Z}$, computing its square and converting it into the same form (as a word of the above letters) by using the commuting relations (3) and (4). Since the squares of $\dot{X}$, $\dot{Y}$, $\dot{Q}$, and $\dot{Z}$ are equal to $\dot{U}$, $\dot{V}$, $\dot{T}$, and $\dot{W}$, respectively, which are the generators of the group ${}^{16}\mathscr{D}^0$, the lemma follows. $\square$

In the proof of the next result we shall see that we have constructed our matrices in such a way that every nondiagonal matrix contains the 2×2 matrices f and g defined in (2) as minor submatrices. This fact determines the spectrum of constructed matrices.

Lemma 6. *The spectrum of every nondiagonal element of* ${}^{16}\mathscr{G}$ *is* $\{1, -1, i, -i\}$.

Proof. Consider the diagonal matrices and observe that the spectrum of all the elements of ${}^{16}\mathscr{D}^0$ is equal to $\{1, -1\}$, each of them with multiplicity 8. Choose A in ${}^{16}\mathscr{G}\backslash{}^{16}\mathscr{D}$. By Lemma 5, A^2 belongs to ${}^{16}\mathscr{D}^0$, which implies that $\sigma(A) \subseteq \{1, -1, i, -i\}$.

Since A is a monomial matrix whose square is diagonal, for every nonzero entry a_{ij} also the entry a_{ji} is nonzero and the product $a_{ij}a_{ji}$ is either 1 or -1. We can assume that $i < j$ and denote by

$$\widetilde{A}_{ij} := \begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix}$$

an (i, j)th minor submatrix of A. Obviously, $\sigma(\widetilde{A}_{ij}) \subseteq \sigma(A)$. Now consider both possibilities: if $a_{ij}a_{ji} = 1$, then both entries a_{ij} and a_{ji} are equal and $\widetilde{A}_{ij}$ is either matrix f or $-f$ from (2). Computing spectrum of f we obtain that

$$\sigma(\widetilde{A}_{ij}) = \sigma(\pm f) = \{1, -1\} \subseteq \sigma(A).$$

In the case when $a_{ij}a_{ji} = -1$, one of the two entries is 1 and the other -1, therefore $\widetilde{A}_{ij}$ is either matrix g or $-g$ from (2). In this case we get

$$\sigma(\widetilde{A}_{ij}) = \sigma(\pm g) = \{i, -i\} \subseteq \sigma(A).$$

Since both possibilities occur, we see that $\{1, -1, i, -i\} \subseteq \sigma(A)$ which completes the proof. □

Proposition 7. *Group* $^{16}\mathscr{G}$ *is an irreducible group of exponent* 4 *with submultiplicative spectrum.*

Proof. First, note that the exponent of the group $^{16}\mathscr{D}$ is equal to 2. From Lemma 5 we conclude that the exponent of the group $^{16}\mathscr{G}$ equals 4.

Let us now examine the spectrum of $^{16}\mathscr{G}$. We have already noticed that the spectrum of diagonal matrices lies in the set $\{1, -1\}$. For arbitrary matrices $A, B \in {}^{16}\mathscr{G}$ submultiplicativity of the spectrum is obvious if they are both diagonal. If one of them, say A, is nondiagonal, then its spectrum is by Lemma 6 maximal possible (that is $\{1, -1, i, -i\}$) and therefore contains the spectrum of the product AB:

$$\sigma(A)\sigma(B) = \{1, -1, i, -i\} \supseteq \sigma(AB).$$

It remains to show irreducibility of the group $^{16}\mathscr{G}$. Suppose that $^{16}\mathscr{G}$ has a nonzero invariant subspace $\mathscr{W}$. Studying actions of matrices $\dot{X}$, $\dot{Y}$, $\dot{Q}$, and $\dot{Z}$ on $\mathscr{W}$ we see that for every i, $1 \leqslant i \leqslant 16$, $\mathscr{W}$ contains an element with nonzero ith coordinate. We give the proof only for the case $i = 1$; similar arguments apply to the other cases. Let w be a nonzero element of $\mathscr{W}$. Since $^{16}\mathscr{G}$ contains $\dot{Z}$ there is no loss of generality in assuming that some of the first 8 coordinates of w are nonzero. The fact that $^{16}\mathscr{G}$ contains $\dot{Q}$ implies that we can further assume that some of the first 4 coordinates of w are nonzero. We can then assume that one of the first two coordinates of w is nonzero, since $^{16}\mathscr{G}$ contains $\dot{Y}$. Now there is no loss of generality in assuming that the first entry of w is nonzero, since $^{16}\mathscr{G}$ also contains $\dot{X}$.

Observe that using actions of diagonal matrices $\dot{U}$, $\dot{V}$, $\dot{T}$, and $\dot{W}$ on $w \in \mathscr{W}$ and applying various linear combinations we can get any standard basis vector. Let us for example take $w = (\alpha_1, \alpha_2, \ldots, \alpha_{16}) \in \mathscr{W}$ with $\alpha_1 \neq 0$ and compute:

$$\begin{aligned}
w_1 &= w + \dot{T}w = (\alpha_1, \ldots, \alpha_8, 0, \ldots, 0),\\
w_2 &= w_1 + \dot{V}w_1 = (\alpha_1, \ldots, \alpha_4, 0, \ldots, 0),\\
w_3 &= w_2 + \dot{U}w_2 = (\alpha_1, \alpha_2, 0, \ldots, 0),\\
w_4 &= w_3 + \dot{W}w_3 = (\alpha_1, 0, \ldots, 0),\\
w_5 &= \tfrac{1}{\alpha_1}w_4 = (1, 0, \ldots, 0).
\end{aligned}$$

In the same manner we can see that $\mathscr{W}$ contains every standard basis vector. Therefore, $\mathscr{W}$ is not a proper subspace and the proof is complete. □

Similar construction is used in the case of 32×32 matrices. We can construct them from already defined 16×16 matrices as follows:

$$\ddot{X} := \dot{X}^{(2)}, \quad \ddot{Y} := \dot{Y}^{(2)}, \quad \ddot{Q} := \dot{Q}^{(2)}, \quad \ddot{P} = \dot{Q}^{(2,2)}, \quad \ddot{Z} := \dot{Z}^{(2,1)}.$$

Then

$$\begin{array}{lll} \ddot{U} := \ddot{X}^2 = \dot{U}^{(2)}, & \ddot{V} := \ddot{Y}^2 = \dot{V}^{(2)}, & \ddot{T} := \ddot{Q}^2 = \dot{T}^{(2)}, \\ \ddot{S} := \ddot{P}^2 = \dot{T}^{(2,1)}, & \ddot{W} := \ddot{Z}^2 = \dot{W}^{(2,1)}. & \end{array}$$

As in the previous case we notice that the diagonal matrices $\ddot{U}$, $\ddot{V}$, $\ddot{T}$, $\ddot{S}$, and $\ddot{W}$ commute with each other while the remaining matrices $\ddot{X}$, $\ddot{Y}$, $\ddot{Q}$, $\ddot{P}$, and $\ddot{Z}$ either commute or anticommute with them:

$$\begin{array}{lll} \ddot{X}\ddot{U} = \ddot{U}\ddot{X}, & \ddot{X}\ddot{V} = \ddot{V}\ddot{X}, & \ddot{X}\ddot{T} = \ddot{T}\ddot{X}, \\ \ddot{Y}\ddot{U} = -\ddot{U}\ddot{Y}, & \ddot{Y}\ddot{V} = \ddot{V}\ddot{Y}, & \ddot{Y}\ddot{T} = \ddot{T}\ddot{Y}, \\ \ddot{Q}\ddot{U} = \ddot{U}\ddot{Q}, & \ddot{Q}\ddot{V} = -\ddot{V}\ddot{Q}, & \ddot{Q}\ddot{T} = \ddot{T}\ddot{Q}, \\ \ddot{P}\ddot{U} = \ddot{U}\ddot{P}, & \ddot{P}\ddot{V} = \ddot{V}\ddot{P}, & \ddot{P}\ddot{T} = -\ddot{T}\ddot{P}, \\ \ddot{Z}\ddot{U} = \ddot{U}\ddot{Z}, & \ddot{Z}\ddot{V} = \ddot{V}\ddot{Z}, & \ddot{Z}\ddot{T} = \ddot{T}\ddot{Z}, \\ & & \\ \ddot{X}\ddot{S} = \ddot{S}\ddot{X}, & \ddot{X}\ddot{W} = -\ddot{W}\ddot{X}, & \\ \ddot{Y}\ddot{S} = \ddot{S}\ddot{Y}, & \ddot{Y}\ddot{W} = \ddot{W}\ddot{Y}, & \\ \ddot{Q}\ddot{S} = \ddot{S}\ddot{Q}, & \ddot{Q}\ddot{W} = -\ddot{W}\ddot{Q}, & \\ \ddot{P}\ddot{S} = \ddot{S}\ddot{P}, & \ddot{P}\ddot{W} = -\ddot{W}\ddot{P}, & \\ \ddot{Z}\ddot{S} = -\ddot{S}\ddot{Z}, & \ddot{Z}\ddot{W} = \ddot{W}\ddot{Z}. & \end{array} \tag{5}$$

It is a simple matter of computation to verify the following commuting relations between $\ddot{X}$, $\ddot{Y}$, $\ddot{Q}$, $\ddot{P}$, and $\ddot{Z}$:

$$\begin{array}{lll} \ddot{X}\ddot{Y} = -\ddot{S}\ddot{W}\ddot{Y}\ddot{X}, & \ddot{Y}\ddot{Q} = \ddot{U}\ddot{Q}\ddot{Y}, & \ddot{Q}\ddot{P} = \ddot{V}\ddot{P}\ddot{Q}, \\ \ddot{X}\ddot{Q} = \ddot{U}\ddot{Q}\ddot{X}, & \ddot{Y}\ddot{P} = \ddot{P}\ddot{Y}, & \ddot{Q}\ddot{Z} = \ddot{U}\ddot{S}\ddot{Z}\ddot{Q}, \\ \ddot{X}\ddot{P} = \ddot{U}\ddot{P}\ddot{X}, & \ddot{Y}\ddot{Z} = -\ddot{W}\ddot{Z}\ddot{Y}, & \ddot{P}\ddot{Z} = \ddot{U}\ddot{T}\ddot{S}\ddot{Z}\ddot{P}, \\ \ddot{X}\ddot{Z} = \ddot{S}\ddot{Z}\ddot{X}. & & \end{array} \tag{6}$$

Matrices $\ddot{X}$, $\ddot{Y}$, $\ddot{Q}$, $\ddot{P}$, and $\ddot{Z}$ generate group ${}^{32}\mathscr{G}$. From the relations (5) and (6) it follows that every element of this group can be expressed as a word in letters $\ddot{U}$, $\ddot{V}$, $\ddot{T}$, $\ddot{S}$, $\ddot{W}$, $\ddot{X}$, $\ddot{Y}$, $\ddot{Q}$, $\ddot{P}$, and $\ddot{Z}$ in a unique way (in given order), with possible negative sign. In particular, ${}^{32}\mathscr{G}$ is a finite group with $2^{11} = 2048$ elements.

Introduce similar notation as before: ${}^{32}\mathscr{D}$ is the set of all diagonal matrices from ${}^{32}\mathscr{G}$ and ${}^{32}\mathscr{D}^0$ is the subset of all elements of ${}^{32}\mathscr{D}$ with trace equal zero. Obviously, ${}^{32}\mathscr{D}$ is a commutative group generated by matrices $\ddot{U}$, $\ddot{V}$, $\ddot{T}$, $\ddot{S}$, $\ddot{W}$, and $-I$, while ${}^{32}\mathscr{D}^0$ consists of diagonal matrices with 16 entries equal to 1 and 16 entries equal to -1. Clearly, group ${}^{32}\mathscr{D}$ contains ${}^{32}\mathscr{D}^0$ together with the matrices I and $-I$. The 32×32 versions of Lemma 5 and Lemma 6 follow by the same method as their 16×16 counterparts, so we omit the proof.

Lemma 8. *The square of every nondiagonal element of* ${}^{32}\mathscr{G}$ *belongs to* ${}^{32}\mathscr{D}^0$.

Lemma 9. *The spectrum of every nondiagonal element of* ${}^{32}\mathscr{G}$ *is* $\{1, -1, i, -i\}$.

The following analogue of Proposition 7 can also be proved in much the same way:

Proposition 10. *Group* ${}^{32}\mathscr{G}$ *is an irreducible group of exponent* 4 *with submultiplicative spectrum.*

We are now in position to prove:

Theorem 11. *For every* $n \in \mathbb{N}$, $n \geqslant 3$, *there exists an irreducible group of* $2^n \times 2^n$ *matrices with submultiplicative spectrum.*

Proof. Our proof starts with observation that for every $n \in \mathbb{N}$, $n \geqslant 3$, there exists $k \in \mathbb{N} \cup \{0\}$ such that $n - 3k$ is contained in $\{3, 4, 5\}$. We thus can write $2^n = 8^k 2^l$ with $l \in \{3, 4, 5\}$ which shows us that we can construct the group we are searching for from the groups ${}^{8}\mathscr{G}$, ${}^{16}\mathscr{G}$, and ${}^{32}\mathscr{G}$:

$$\mathscr{G} = {}^{8}\mathscr{G}^{\otimes k} \otimes {}^{2^l}\mathscr{G}$$

(with $\mathscr{H}^{\otimes k}$ we denote the tensor product of k copies of group $\mathscr{H}$). Recall that irreducibility and submultiplicativity of spectrum are invariant under the tensor product of groups (Proposition 3) and the proof is complete. □

Let us remark, that the theorem is not valid for $n = 1, 2$, as we will see in Proposition 13.

Examples of irreducible groups of $p \times p$ matrices with submultiplicative spectrum, where p is any odd prime, can be found in [4, Example 3.3.7]. We will denote these groups by ${}^{p}\mathscr{G}$. They are generated by a cyclic permutation of order p and a group ${}^{p}\mathscr{D}$ of diagonal matrices of determinant 1 whose eigenvalues are all distinct pth roots of unity. Since every nondiagonal element of ${}^{p}\mathscr{G}$ is similar to a cyclic permutation of order p, we get $|{}^{p}\mathscr{G}| = p \cdot |{}^{p}\mathscr{D}|$. It is easily seen that we can choose ${}^{p}\mathscr{D}$ with order p^2, hence $|{}^{p}\mathscr{G}| = p^3$. Combining these examples with Proposition 3 gives examples of irreducible groups with submultiplicative spectrum over all possible odd-dimensional spaces [4, Theorem 3.3.8]. Applying previously constructed groups we obtain a generalization of these examples by the same method.

Corollary 12. *For every even m which is divisible by* 8 *there exists an irreducible group of* $m \times m$ *matrices with submultiplicative spectrum.*

Proof. Let $m = 2^n k$ where $n \geqslant 3$ and k odd. If $\mathscr{G}$ is the group of $2^n \times 2^n$ matrices from Theorem 11 and $\mathscr{H}$ group of $k \times k$ matrices constructed in [4, Theorem 3.3.8] then $\mathscr{G} \otimes \mathscr{H}$ is an irreducible group of $m \times m$ matrices with submultiplicative spectrum. □

The assumption on divisibility by 8 is necessary (cf. Corollary 14).

3. Minimality

In this section we discuss various aspects of minimality of examples constructed in the previous section. We first show that 8 is the minimal possible dimension of matrices. The case of dimension 2 was already mentioned in [2]. Our proof is based on the results of [3].

Proposition 13. *Every group of* 2×2 *or* 4×4 *matrices with submultiplicative spectrum is reducible.*

Proof. Let us first consider the case of 2×2 matrices. Suppose $\mathscr{G}$ is an irreducible group of 2×2 matrices with submultiplicative spectrum and exponent 2^{s+1}. We can assume that $s > 0$, since otherwise we are dealing with the group of involutions which is commutative and reducible. Hence the length of series (1) for $\mathscr{G}$ is at least 3. It follows from Lemma 4 that the group $\mathscr{G}$ is monomial and $\mathscr{G}_s$ diagonal. Every $A \in \mathscr{G}\backslash\mathscr{G}_s$ has maximal exponent (i.e. 2^{s+1}). Since A is a monomial nondiagonal 2×2 matrix, its square is a scalar matrix. There is no loss of generality in assuming that $A^2 = I$ (otherwise we replace $\mathscr{G}$ by $-(\det A)^{-1}\mathscr{G}$ which does not affect neither the irreducibility of the group nor the submultiplicativity of its spectrum). Therefore the exponent of the whole group is 2 and the group is reducible. Note that we have actually used only the property (1) of the group $\mathscr{G}$ and not the submultiplicativity of its spectrum.

If $\mathscr{G}$ is an irreducible group of 4×4 matrices with submultiplicative spectrum, it follows from Lemma 4 that the group $\mathscr{G}$ is block-monomial and $\mathscr{G}_s$ is block-diagonal with irreducible diagonal blocks. First observe that the dimension 2 of these blocks is not possible, since the restrictions of $\mathscr{G}$ to first diagonal block would then be an irreducible group of 2×2 matrices with normal series (1). Thus the group $\mathscr{G}$ is actually monomial and $\mathscr{G}_s$ is diagonal. We see by the same method as above that then the exponent of $\mathscr{G}$ is 4 (as before we may assume that the exponent is not 2). Choose $A \in \mathscr{G}$ such that $A^4 = I$ and $A^2 = D \neq \pm I$. Using monomiality of A it is easy to check that $\det A = -1$. By [3, Theorem 4.5] the group $\mathscr{G}$ is then reducible. $\square$

Corollary 14. *If m is an even integer which is not divisible by* 8, *then every group of* $m \times m$ *matrices with submultiplicative spectrum is reducible.*

Proof. Let $m = 2^n k$ where $n \in \{1, 2\}$ and k odd. Suppose, contrary to our claim, that there exists an irreducible group $\mathscr{G}$ with submultiplicative spectrum, consisting of $m \times m$ matrices. Then m divides the order of the group $\mathscr{G}$ (cf. [6, § 6.5, Corollary 2]) which shows that $\mathscr{G}$ contains a Sylow 2-subgroup $\mathscr{H}$. Combining Propositions 1 and 2 we conclude that $\mathscr{H}$ is also irreducible, which is impossible, since $\mathscr{H}$ consists of 2×2 or 4×4 matrices. $\square$

The exponent of our groups is minimal possible. We have already noticed that any group with exponent 2 consists of involutions and is therefore reducible. Since the exponent of the groups ${}^{8}\mathscr{G}$, ${}^{16}\mathscr{G}$, and ${}^{32}\mathscr{G}$ is 4, it is minimal. We check at once that for any odd prime p the exponent of the group ${}^{p}\mathscr{G}$ equals p which is also minimal. Hence the groups constructed in the proof of Corollary 12 have minimal possible exponent.

More important, our groups have minimal possible order. For any prime p let $\mathscr{G}$ be an irreducible p-group of $n \times n$ matrices where $n = p^k$ for some $k \in \mathbb{N}$. From Burnside's Theorem (cf. [4, Theorem 1.2.2]) it follows that $\mathscr{G}$ contains n^2 linearly independent elements (over $\mathbb{C}$). We know that p divides the order of the center $Z(\mathscr{G})$ of our p-group $\mathscr{G}$ (cf. [5, Theorem 4.4]). Since the center $Z(\mathscr{G})$ consists of scalar matrices, we conclude that

$$|\mathscr{G}| \geqslant n^2|Z(\mathscr{G})| \geqslant n^2 p. \tag{7}$$

Observe that in the case of groups ${}^{8}\mathscr{G}$, ${}^{16}\mathscr{G}$, ${}^{32}\mathscr{G}$, and ${}^{p}\mathscr{G}$ we get equality in (7) and therefore the order of these groups is minimal. Proposition 1 implies that also the groups constructed in the proof of Corollary 12 have minimal order.

Remark 15. The groups for which equality in (7) occurs are known as *groups of central type*. These are finite groups with maximal possible degree of irreducible representation. They are all solvable but not necessary nilpotent (see for instance [1]). Groups constructed in the present article are examples of nilpotent groups of central type.

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References

[1] R.B. Howlett, I.M. Isaacs, On groups of central type, Math Z. 179 (1982) 555–569.

[2] M. Lambrou, W.E. Longstaff, H. Radjavi, Spectral conditions and reducibility of operator semigroups, Indiana Univ. Math. J. 41 (1992) 449–545.

[3] M. Omladič, On 2-groups with submultiplicative spectrum, J. Pure Appl. Algebra 167 (2002) 315–328.

[4] H. Radjavi, P. Rosenthal, Simultaneous Triangularization, Springer-Verlag, New York, 2000.

[5] J.J. Rotman, An Introduction to the Theory of Groups, Wm. C. Brown Publishers, Dubuque, 1988.

[6] J.-P. Serre, Linear Representations of Finite Groups, GTM 42, Springer-Verlag, New York, 1977.